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The natural transformations between r -th order prolongation of tangent and cotangent bundles over Riemannian manifolds

ABSTRACT. If (M, g) is a Riemannian manifold then there is the well-known base preserving vector bundle isomorphism $TM \rightarrow T^*M$ given by $v \mapsto g(v, -)$ between the tangent TM and the cotangent T^*M bundles of M . In the present note first we generalize this isomorphism to the one $J^r TM \rightarrow J^r T^*M$ between the r -th order prolongation $J^r TM$ of tangent TM and the r -th order prolongation $J^r T^*M$ of cotangent T^*M bundles of M . Further we describe all base preserving vector bundle maps $D_M(g): J^r TM \rightarrow J^r T^*M$ depending on a Riemannian metric g in terms of natural (in g) tensor fields on M .

1. Introduction. All manifolds are smooth, Hausdorff, finite dimensional and without boundaries. Maps are assumed to be smooth, i.e. of class C^∞ . Let $\mathcal{M}f_m$ denote category of m -dimensional manifolds and their embeddings.

From the general theory it is well known that the tangent TM and the cotangent T^*M bundles of M are not canonically isomorphic. However, if g is a Riemannian metric on a manifold M , there is the base preserving vector bundle isomorphism $i_g: TM \rightarrow T^*M$ given by $i_g(v) = g(v, -), v \in T_x M, x \in M$.

In the second section of the present note we give necessary definitions.

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In the third section first we generalize the isomorphism $i_g: TM \rightarrow T^*M$ depending on g to a base preserving vector bundle isomorphism $J^r i_g: J^r TM \rightarrow J^r T^*M$ canonically depending on g between the r -th order prolongation $J^r TM$ of tangent TM and the r -th order prolongation $J^r T^*M$ of cotangent T^*M bundles of M . Next we construct another more advanced base preserving vector bundle isomorphism $i_g^{<r>}: J^r TM \rightarrow J^r T^*M$ canonically depending on g .

In the fourth section we consider the problem of describing all $\mathcal{M}f_m$ -natural operators $D: Riem \rightsquigarrow Hom(J^r T, J^r T^*)$ transforming Riemannian metrics g on m -dimensional manifolds M into base preserving vector bundle maps $D_M(g): J^r TM \rightarrow J^r T^*M$. Our studies lead to the reduction of this problem to the one of describing all $\mathcal{M}f_m$ -natural operators $t: Riem \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ (for $l, k = 1, \dots, r$) sending Riemannian metrics g on M into tensor fields $t_M(g)$ of types $T^* \otimes S^l T \otimes T^* \otimes S^k T^*$.

2. Definitions. Now we give some necessary definitions.

Definition 1. The r -th order prolongation of tangent bundle is a functor $J^r T: \mathcal{M}f_m \rightarrow \mathcal{VB}$ sending any m -manifold M into $J^r TM$ and any embedding $\varphi: M_1 \rightarrow M_2$ of two manifolds into $J^r T\varphi: J^r TM_1 \rightarrow J^r TM_2$ given by $J^r T\varphi(j_x^r X) = j_{\varphi(x)}^r \varphi_* X$, where $X \in \mathcal{X}(M_1)$ and $\varphi_* X = T\varphi \circ X \circ \varphi^{-1}$ is the image of a vector field X by φ .

Definition 2. The r -th order prolongation of cotangent bundle is a functor $J^r T^*: \mathcal{M}f_m \rightarrow \mathcal{VB}$ sending any m -manifold M into $J^r T^*M$ and any embedding $\varphi: M_1 \rightarrow M_2$ of two manifolds into

$$J^r T^* \varphi: J^r T^* M_1 \rightarrow J^r T^* M_2$$

given by $J^r T^* \varphi := J^r (T\varphi^{-1})^*$.

Definition 3. The dual bundle of the r -th order prolongation of tangent bundle is a functor $(J^r T)^*: \mathcal{M}f_m \rightarrow \mathcal{VB}$ sending any m -manifold M into $(J^r T)^* M := (J^r TM)^*$ and any embedding $\varphi: M_1 \rightarrow M_2$ of two manifolds into

$$(J^r T)^* \varphi: (J^r T)^* M_1 \rightarrow (J^r T)^* M_2$$

given by $(J^r T)^* \varphi := (J^r T\varphi^{-1})^*$.

Definition 4. The dual bundle of the r -th order prolongation of cotangent bundle is a functor $(J^r T^*)^*: \mathcal{M}f_m \rightarrow \mathcal{VB}$ sending any m -manifold M into $(J^r T^*)^* M := (J^r T^* M)^*$ and any embedding $\varphi: M_1 \rightarrow M_2$ of two manifolds into

$$(J^r T^*)^* \varphi: (J^r T^*)^* M_1 \rightarrow (J^r T^*)^* M_2$$

given by $(J^r T^*)^* \varphi := (J^r T^* \varphi^{-1})^*$.

The general concept of natural operators can be found in [4]. In particular, we have the following definitions.

Definition 5. An $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^rT, J^rT^*)$ transforming Riemannian metrics g on m -dimensional manifolds M into base preserving vector bundle maps $D_M(g): J^rTM \rightarrow J^rT^*M$ is a system $D = \{D_M\}_{M \in Obj(\mathcal{M}f_m)}$ of regular operators

$$D_M: Riem(M) \rightarrow Hom_M(J^rTM, J^rT^*M)$$

satisfying the $\mathcal{M}f_m$ -invariance condition, where $Hom_M(J^rTM, J^rT^*M)$ is the set of all vector bundle maps $J^rTM \rightarrow J^rT^*M$ covering the identity map id_M of M .

The $\mathcal{M}f_m$ -invariance condition of D is following: for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ -related by an embedding $\varphi: M_1 \rightarrow M_2$ of m -manifolds (i.e. φ is (g_1, g_2) -isomorphism) then $D_{M_1}(g_1)$ and $D_{M_2}(g_2)$ are also φ -related (i.e. $D_{M_2}(g_2) \circ J^rT\varphi = J^rT^*\varphi \circ D_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$ -invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram

$$(1) \quad \begin{array}{ccc} T^*M_1 \otimes T^*M_1 & \xrightarrow{T^*\varphi \otimes T^*\varphi} & T^*M_2 \otimes T^*M_2 \\ \uparrow g_1 & & \uparrow g_2 \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

commutes for an embedding $\varphi: M_1 \rightarrow M_2$ (i.e. $(T^*\varphi \otimes T^*\varphi) \circ g_1 = g_2 \circ \varphi$) then the diagram

$$\begin{array}{ccc} J^rT^*M_1 & \xrightarrow{J^rT^*\varphi} & J^rT^*M_2 \\ \uparrow D_{M_1}(g_1) & & \uparrow D_{M_2}(g_2) \\ J^rTM_1 & \xrightarrow{J^rT\varphi} & J^rTM_2 \end{array}$$

commutes also.

We say that operator D_M is regular if it transforms smoothly parameterized families of Riemannian metrics into smoothly parameterized ones of vector bundle maps.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^rT, J^rT)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^rT, (J^rT)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^rT, (J^rT^*)^*)$,

- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^r T^*, J^r T)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^r T^*, J^r T^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^r T^*, (J^r T)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom(J^r T^*, (J^r T^*)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T)^*, J^r T)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T)^*, J^r T^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T)^*, (J^r T)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T)^*, (J^r T^*)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T^*)^*, J^r T)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T^*)^*, J^r T^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T^*)^*, (J^r T)^*)$,
- an $\mathcal{M}f_m$ -natural operator $D: Riem \rightsquigarrow Hom((J^r T^*)^*, (J^r T^*)^*)$.

Now we have the following definition.

Definition 6. An $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T^*, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m -dimensional manifolds M into base preserving vector bundle maps $A_M(g): TM \otimes S^l T^* M \rightarrow T^* M \otimes S^k T^* M$ is a system $A = \{A_M\}_{M \in \text{Obj}(\mathcal{M}f_m)}$ of regular operators $A_M: Riem(M) \rightarrow C^\infty(TM \otimes S^l T^* M, T^* M \otimes S^k T^* M)$ satisfying the $\mathcal{M}f_m$ -invariance condition, where $C^\infty(TM \otimes S^l T^* M, T^* M \otimes S^k T^* M)$ is the set of all vector bundle maps $TM \otimes S^l T^* M \rightarrow T^* M \otimes S^k T^* M$ covering the identity map id_M of M .

The $\mathcal{M}f_m$ -invariance condition of A is following : for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ -related by an embedding $\varphi: M_1 \rightarrow M_2$ of m -manifolds (i.e. $(T^* \varphi \otimes T^* \varphi) \circ g_1 = g_2 \circ \varphi$) then $A_{M_1}(g_1)$ and $A_{M_2}(g_2)$ are also φ -related (i.e. $A_{M_2}(g_2) \circ (T \varphi \otimes S^l T^* \varphi) = (T^* \varphi \otimes S^k T^* \varphi) \circ A_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$ -invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram (1) commutes for an embedding $\varphi: M_1 \rightarrow M_2$ then the diagram

$$\begin{array}{ccc}
 T^* M_1 \otimes S^k T^* M_1 & \xrightarrow{T^* \varphi \otimes S^k T^* \varphi} & T^* M_2 \otimes S^k T^* M_2 \\
 \uparrow A_{M_1}(g_1) & & \uparrow A_{M_2}(g_2) \\
 TM_1 \otimes S^l T^* M_1 & \xrightarrow{T \varphi \otimes S^l T^* \varphi} & TM_2 \otimes S^l T^* M_2
 \end{array}$$

commutes also.

The regularity means almost the same as in Definition 5.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T, T \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T)$,

- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T^*, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T^*)$,
- an $\mathcal{M}f_m$ -natural operator $A: Riem \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T^*)$.

Next we have an important general definition of natural tensor.

Definition 7. An $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ transforming Riemannian metrics g on m -dimensional manifolds M into tensor fields of type (p, q) on M is a system $t = \{t_M\}_{M \in \text{Obj}(\mathcal{M}f_m)}$ of regular operators $t_M: Riem(M) \rightarrow \mathcal{T}^{(p,q)}(M)$ satisfying the $\mathcal{M}f_m$ -invariance condition, where $\mathcal{T}^{(p,q)}(M)$ is the set of tensor fields of type (p, q) on M .

The $\mathcal{M}f_m$ -invariance condition of t is following : for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if g_1 and g_2 are φ -related by an embedding $\varphi: M_1 \rightarrow M_2$ of m -manifolds (i.e. $(T^* \varphi \otimes T^* \varphi) \circ g_1 = g_2 \circ \varphi$) then $t_{M_1}(g_1)$ and $t_{M_2}(g_2)$ are also φ -related (i.e. $t_{M_2}(g_2) \circ \varphi = (\bigotimes^p T \varphi \otimes \bigotimes^q T^* \varphi) \circ t_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$ -invariance means that for any $g_1 \in Riem(M_1)$ and $g_2 \in Riem(M_2)$ if the diagram (1) commutes for an embedding $\varphi: M_1 \rightarrow M_2$, then the diagram

$$\begin{array}{ccc}
 \bigotimes^p T M_1 \otimes \bigotimes^q T^* M_1 & \xrightarrow{\bigotimes^p T \varphi \otimes \bigotimes^q T^* \varphi} & \bigotimes^p T M_2 \otimes \bigotimes^q T^* M_2 \\
 \uparrow t_{M_1}(g_1) & & \uparrow t_{M_2}(g_2) \\
 M_1 & \xrightarrow{\varphi} & M_2
 \end{array}$$

commutes also.

We say that operator t_M is regular if it transforms smoothly parametrized families of Riemannian metrics into smoothly parametrized ones of tensor fields.

Now we have a definition of a special kind of natural tensor.

Definition 8. An $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -dimensional manifolds M into tensor fields of type $T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ on M is a system $t = \{t_M\}_{M \in \text{Obj}(\mathcal{M}f_m)}$ of regular operators $t_M: \text{Riem}(M) \rightarrow C^\infty(T^*M \otimes S^l TM \otimes T^*M \otimes S^k T^*M)$ satisfying the $\mathcal{M}f_m$ -invariance condition, where $C^\infty(T^*M \otimes S^l TM \otimes T^*M \otimes S^k T^*M)$ is the set of all tensor fields of type $T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ on M .

The $\mathcal{M}f_m$ -invariance condition of t is following: for any $g_1 \in \text{Riem}(M_1)$ and $g_2 \in \text{Riem}(M_2)$ if g_1 and g_2 are φ -related by an embedding $\varphi: M_1 \rightarrow M_2$ of m -manifolds (i.e. $(T^*\varphi \otimes T^*\varphi) \circ g_1 = g_2 \circ \varphi$), then $t_{M_1}(g_1)$ and $t_{M_2}(g_2)$ are also φ -related (i.e. $t_{M_2}(g_2) \circ \varphi = (T^*\varphi \otimes S^l T\varphi \otimes T^*\varphi \otimes S^k T^*\varphi) \circ t_{M_1}(g_1)$).

Equivalently, the above $\mathcal{M}f_m$ -invariance means that for any $g_1 \in \text{Riem}(M_1)$ and $g_2 \in \text{Riem}(M_2)$ if the diagram (1) commutes for an embedding $\varphi: M_1 \rightarrow M_2$, then the diagram

$$\begin{array}{ccc}
 T^*M_1 \otimes S^l TM_1 \otimes T^*M_1 \otimes S^k T^*M_1 & \xrightarrow{\Phi} & T^*M_2 \otimes S^l TM_2 \otimes T^*M_2 \otimes S^k T^*M_2 \\
 \uparrow t_{M_1}(g_1) & & \uparrow t_{M_2}(g_2) \\
 M_1 & \xrightarrow{\varphi} & M_2
 \end{array}$$

commutes also, where $\Phi = T^*\varphi \otimes S^l T\varphi \otimes T^*\varphi \otimes S^k T^*\varphi$.

The regularity means almost the same as in Definition 7.

Similarly, we can define the following concepts:

- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T \otimes S^l T^* \otimes T \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T^* \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T^* \otimes S^l T^* \otimes T \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T$,

- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T^* \otimes S^l T \otimes T \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T \otimes S^l T^* \otimes T \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T \otimes S^l T \otimes T^* \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T^* \otimes S^l T^* \otimes T^* \otimes S^k T$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T^* \otimes S^l T^* \otimes T \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T^*$,
- an $\mathcal{M}f_m$ -natural operator (natural tensor) $t: Riem \rightsquigarrow T^* \otimes S^l T^* \otimes T^* \otimes S^k T^*$.

In the third section we present also explicit examples of $\mathcal{M}f_m$ -natural operators $D: Riem \rightsquigarrow Hom(J^r T, J^r T^*)$.

A full description of all polynomial natural tensors $t: Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ transforming Riemannian metrics on m -manifolds into tensor fields of types (p, q) can be found in [1]. This description is following. Each covariant derivative of the curvature $\mathcal{R}(g) \in \mathcal{T}^{(0,4)}(M)$ of a Riemannian metric g is a natural tensor and the metric g is also a natural tensor. Further all the natural tensors $t: Riem \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ can be obtained by a procedure:

- (a) every tensor multiplication of two natural tensors give a new natural tensor,
- (b) every contraction on one covariant and one contravariant entry of a natural tensor give a new natural tensor,
- (c) we can tensorize any natural tensor with a metric independent natural tensor,
- (d) we can permute any number of entries in the tensor product,
- (e) we can repeat these steps,
- (f) we can take linear combinations.

Furthermore, if we take respective type natural tensors and apply respective symmetrization, then we can produce many natural tensors $t: Riem \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T^*$.

3. Constructions.

Example 1. Let (M, g) be a Riemannian manifold. Then we have a base preserving vector bundle isomorphism $i_g: TM \rightarrow T^*M$ given by

$$i_g(v) = g(v, -), \quad v \in T_x M, \quad x \in M.$$

Next we can obtain a base preserving vector bundle isomorphism $J^r i_g: J^r TM \rightarrow J^r T^*M$ defined by a formula

$$J^r i_g(j_x^r X) = j_x^r(i_g \circ X),$$

where $X \in \mathcal{X}(M)$. Similarly we receive also a base preserving vector bundle isomorphism

$$(J^r i_g^{-1})^*: (J^r TM)^* \rightarrow (J^r T^*M)^*.$$

Because of the canonical character of the above constructions we get the following propositions.

Proposition 1. *The family $A^{(r)}: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, J^r T^*)$ of operators*

$$A_M^{(r)}: \text{Riem}(M) \rightarrow \text{Hom}_M(J^r TM, J^r T^*M), \quad A_M^{(r)}(g) = J^r i_g$$

for all $M \in \text{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 2. *The family $A^{[r]}: \text{Riem} \rightsquigarrow \text{Hom}((J^r T)^*, (J^r T^*)^*)$ of operators*

$$A_M^{[r]}: \text{Riem}(M) \rightarrow \text{Hom}_M((J^r TM)^*, (J^r T^*M)^*), \quad A_M^{[r]}(g) = (J^r i_g^{-1})^*$$

for all $M \in \text{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Now we are going to present another more advanced example of an $\mathcal{M}f_m$ -natural operator $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, J^r T^*)$.

Recall that if g is a Riemannian tensor field on a manifold M and $x \in M$, then there is g -normal coordinate system $\varphi: (M, x) \rightarrow (\mathbb{R}^m, 0)$ with centre x . If $\psi: (M, x) \rightarrow (\mathbb{R}^m, 0)$ is another g -normal coordinate system with centre x , then there is $A \in O(m)$ such that $\psi = A \circ \varphi$ near x . Let $I: J_0^r T \mathbb{R}^m \rightarrow \bigoplus_{k=1}^r T_0 \mathbb{R}^m \otimes S^k T_0^* \mathbb{R}^m = \bigoplus_{k=0}^r \mathbb{R}^m \otimes S^k \mathbb{R}^{m*}$ (see [3]) and $I_1: J_0^r T^* \mathbb{R}^m \rightarrow \bigoplus_{k=1}^r T_0^* \mathbb{R}^m \otimes S^k T_0 \mathbb{R}^m = \bigoplus_{k=0}^r \mathbb{R}^{m*} \otimes S^k \mathbb{R}^m$ (see [7]) be the standard $O(m)$ -invariant vector space isomorphisms.

We have the following important proposition.

Proposition 3. *Let g be a Riemannian tensor field on a manifold M . Then there are (canonical in g) vector bundle isomorphisms*

$$I_g: J^r TM \rightarrow \bigoplus_{k=0}^r TM \otimes S^k T^*M,$$

$$J_g: J^r T^*M \rightarrow \bigoplus_{k=0}^r T^*M \otimes S^k T^*M,$$

$$(I_g^{-1})^*: (J^r TM)^* \rightarrow \left(\bigoplus_{k=0}^r TM \otimes S^k T^*M \right)^* \cong \bigoplus_{k=0}^r T^*M \otimes S^k TM,$$

$$(J_g^{-1})^*: (J^r T^*M)^* \rightarrow \left(\bigoplus_{k=0}^r T^*M \otimes S^k T^*M \right)^* \cong \bigoplus_{k=0}^r TM \otimes S^k TM.$$

Proof. Let $v = j_x^r X \in J_x^r TM$, where $X \in \mathcal{X}(M)$, $x \in M$. Let $\varphi: (M, x) \rightarrow (\mathbb{R}^m, 0)$ be a g -normal coordinate system with centre x . We define

$$I_g(v) := I_g^\varphi(v) = \left(\bigoplus_{k=0}^r T\varphi^{-1} \otimes S^k T^* \varphi^{-1} \right) \circ I \circ J^r T \varphi(v).$$

If $\psi: (M, x) \rightarrow (\mathbb{R}^m, 0)$ is another g -normal coordinate system with centre x , then $\psi = A \circ \varphi$ (near x) for some $A \in O(m)$. The $O(m)$ -invariance of I means that

$$(2) \quad I \circ J^r T A = \left(\bigoplus_{k=0}^r T_0 A \otimes S^k T_0^* A \right) \circ I.$$

Hence we deduce that

$$\begin{aligned} I_g^\psi(v) &= \left(\bigoplus_{k=0}^r T\psi^{-1} \otimes S^k T^* \psi^{-1} \right) \circ I \circ J^r T \psi(v) \\ &= \bigoplus_{k=0}^r (T(A \circ \varphi)^{-1} \otimes S^k T^*(A \circ \varphi)^{-1}) \circ I \circ J^r T(A \circ \varphi)(v) \\ &= \bigoplus_{k=0}^r ((T\varphi^{-1} \circ T A^{-1}) \otimes S^k T^*(\varphi^{-1} \circ A^{-1})) \circ I \circ (J^r T A \circ J^r T \varphi)(v) \\ &= \bigoplus_{k=0}^r ((T\varphi^{-1} \circ T A^{-1}) \otimes S^k T^*(\varphi^{-1} \circ A^{-1})) \circ (I \circ J^r T A) \circ J^r T \varphi(v) =: L. \end{aligned}$$

Now using (2), we receive

$$\begin{aligned} L &= \bigoplus_{k=0}^r ((T\varphi^{-1} \circ T A^{-1}) \otimes S^k T^*(\varphi^{-1} \circ A^{-1})) \circ \left(\bigoplus_{k=0}^r T A \otimes S^k T^* A \right) \circ I \circ J^r T \varphi(v) \\ &= \bigoplus_{k=0}^r \left[((T\varphi^{-1} \circ T A^{-1}) \circ T A) \otimes (S^k T^*(\varphi^{-1} \circ A^{-1}) \circ S^k T^* A) \right] \circ I \circ J^r T \varphi(v) \\ &= \bigoplus_{k=0}^r ((T\varphi^{-1} \circ T A^{-1} \circ T A) \otimes S^k T^*(\varphi^{-1} \circ A^{-1} \circ A)) \circ I \circ J^r T \varphi(v) \\ &= \bigoplus_{k=0}^r (T\varphi^{-1} \otimes S^k T^* \varphi^{-1}) \circ I \circ J^r T \varphi(v) = I_g^\varphi(v). \end{aligned}$$

Therefore, the definition of $I_g(v)$ is independent of the choice of φ . So, isomorphism $I_g: J^r TM \rightarrow \bigoplus_{k=0}^r TM \otimes S^k T^* M$ is well defined.

Similarly, we put

$$J_g(v) := J_g^\varphi(v) = \left(\bigoplus_{k=0}^r T^* \varphi^{-1} \otimes S^k T^* \varphi^{-1} \right) \circ I_1 \circ J^r T^* \varphi(v).$$

Using $O(m)$ -invariance of I_1 (i.e. $I_1 \circ J^r T^* A = (\bigoplus_{k=0}^r T_0^* A \otimes S^k T_0^* A) \circ I_1$) analogously as before, we show that $I_g^\psi(v) = I_g^\varphi(v)$. This proves that the definition of $J_g(v)$ is independent of the choice of g -normal coordinate system φ with centre x and the isomorphism $J_g: J^r T^* M \rightarrow \bigoplus_{k=0}^r T^* M \otimes S^k T^* M$ is well defined.

Finally we obtain (canonical in g) vector bundle isomorphisms

$$\begin{aligned} (I_g^{-1})^*: (J^r TM)^* &\rightarrow \left(\bigoplus_{k=0}^r TM \otimes S^k T^* M \right)^* \cong \bigoplus_{k=0}^r T^* M \otimes S^k TM \\ (J_g^{-1})^*: (J^r T^* M)^* &\rightarrow \left(\bigoplus_{k=0}^r T^* M \otimes S^k T^* M \right)^* \cong \bigoplus_{k=0}^r TM \otimes S^k TM. \quad \square \end{aligned}$$

Remark 1. W. Mikulski (in [7]) has recently constructed a (canonical in ∇) vector bundle isomorphism $I_\nabla: J^r TM \rightarrow \bigoplus_{k=0}^r T^* M \otimes S^k T^* M$ for a classical linear connection ∇ on a manifold M .

Now we have further important identifications.

Example 2. Let (M, g) be a Riemannian manifold and $i_g: TM \rightarrow T^* M$ be a base preserving vector bundle isomorphism recalled in Example 1. Using the base preserving vector bundle isomorphisms I_g and J_g from Proposition 3, we receive the following vector bundle isomorphisms

$$\begin{aligned} i_g^{<r>} &:= J_g^{-1} \circ \left(\bigoplus_{k=0}^r i_g \otimes S^k T^* id_M \right) \circ I_g: J^r TM \rightarrow J^r T^* M, \\ i_g^{[r]} &:= J_g^* \circ \left(\bigoplus_{k=0}^r i_g^{-1} \otimes S^k T id_M \right) \circ (I_g^{-1})^*: (J^r TM)^* \rightarrow (J^r T^* M)^*. \end{aligned}$$

Because of canonical character of above constructions we obtain the following propositions.

Proposition 4. The family $B^{<r>}: Riem \rightsquigarrow Hom(J^r T, J^r T^*)$ of operators

$$B_M^{<r>}: Riem(M) \rightarrow Hom_M(J^r TM, J^r T^* M), \quad B_M^{<r>}(g) = i_g^{<r>}$$

for all $M \in obj(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 5. The family $B^{[r]}: Riem \rightsquigarrow Hom((J^r T)^*, (J^r T^*)^*)$ of operators

$$B_M^{[r]}: Riem(M) \rightarrow Hom_M((J^r TM)^*, (J^r T^* M)^*), \quad B_M^{[r]}(g) = i_g^{[r]}$$

for all $M \in obj(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Example 3. Let (M, g) be a Riemannian manifold. In an article [5] J. Kurek and W. Mikulski constructed a base preserving vector bundle isomorphism

$$i_g^{(r)}: \bigoplus_{k=1}^r S^k TM \rightarrow \bigoplus_{k=1}^r S^k T^* M$$

given by

$$i_g^{(r)}(v_1 \odot \cdots \odot v_k) = i_g(v_1) \odot \cdots \odot i_g(v_k).$$

Now using this isomorphism, we get a base preserving vector bundle isomorphism

$$\begin{aligned} I_g^{(r)}: TM \otimes \bigoplus_{k=0}^r S^k T^* M &\cong \bigoplus_{k=0}^r TM \otimes S^k T^* M \\ &\rightarrow T^* M \otimes \bigoplus_{k=0}^r S^k TM \cong \bigoplus_{k=0}^r T^* M \otimes S^k TM \end{aligned}$$

defined by a formula

$$I_g^{(r)} = i_g \otimes (i_g^{(r)})^{-1}.$$

Similarly, we construct another base preserving vector bundle isomorphisms

$$\begin{aligned} \tilde{I}_g^{(r)}: \bigoplus_{k=0}^r TM \otimes S^k T^* M &\rightarrow \bigoplus_{k=0}^r TM \otimes S^k TM, \quad \tilde{I}_g^{(r)} = id_{TM} \otimes (i_g^{(r)})^{-1}, \\ \tilde{\tilde{I}}_g^{(r)}: \bigoplus_{k=0}^r T^* M \otimes S^k T^* M &\rightarrow \bigoplus_{k=0}^r T^* M \otimes S^k TM, \quad \tilde{\tilde{I}}_g^{(r)} = i_g^{-1} \otimes (i_g^{(r)})^{-1}, \\ \hat{I}_g^{(r)}: \bigoplus_{k=0}^r T^* M \otimes S^k T^* M &\rightarrow \bigoplus_{k=0}^r T^* M \otimes S^k TM, \quad \hat{I}_g^{(r)} = id_{T^* M} \otimes (i_g^{(r)})^{-1}. \end{aligned}$$

Thus we receive a base preserving vector bundle isomorphism

$$I_g^{<r>}: J^r TM \rightarrow (J^r TM)^*$$

given by

$$I_g^{<r>} = I_g^* \circ I_g^{(r)} \circ I_g.$$

Similarly, we construct another base preserving vector bundle isomorphisms

$$\begin{aligned} \tilde{I}_g^{<r>}: J^r TM &\rightarrow (J^r T^* M)^*, \quad \tilde{I}_g^{<r>} = J_g^* \circ \tilde{I}_g^{(r)} \circ I_g, \\ I_g^{[r]}: J^r T^* M &\rightarrow (J^r TM)^*, \quad I_g^{[r]} = I_g^* \circ \hat{I}_g^{(r)} \circ J_g, \\ \tilde{\tilde{I}}_g^{[r]}: J^r T^* M &\rightarrow (J^r T^* M)^*, \quad \tilde{\tilde{I}}_g^{[r]} = J_g^* \circ \tilde{\tilde{I}}_g^{(r)} \circ J_g. \end{aligned}$$

Using the base preserving vector bundle isomorphism $J^r i_g: J^r TM \rightarrow J^r T^* M$ constructed in Example 1, we obtain also the following vector bundle isomorphisms

$$\begin{aligned} J_g^{<r>} &= (J^r i_g^{-1})^* \circ I_g^{<r>}: J^r TM \rightarrow (J^r T^* M)^*, \\ \tilde{J}_g^{<r>} &= (J^r i_g)^* \circ \tilde{I}_g^{<r>}: J^r TM \rightarrow (J^r TM)^*, \\ J_g^{[r]} &= (J^r i_g^{-1})^* \circ I_g^{[r]}: J^r T^* M \rightarrow (J^r T^* M)^*, \\ \tilde{\tilde{J}}_g^{[r]} &= (J^r i_g)^* \circ \tilde{\tilde{I}}_g^{[r]}: J^r T^* M \rightarrow (J^r TM)^*. \end{aligned}$$

Because of canonical character of the above constructions we get the following propositions.

Proposition 6. *The family $C^{<r>}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT, (J^rT)^*)$ of operators*

$C_M^{<r>}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rTM, (J^rTM)^*), \quad C_M^{<r>}(g) = I_g^{<r>}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 7. *The family $\tilde{C}^{<r>}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT, (J^rT^*)^*)$ of operators*

$\tilde{C}_M^{<r>}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rTM, (J^rT^*M)^*), \quad \tilde{C}_M^{<r>}(g) = \tilde{I}_g^{<r>}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 8. *The family $C^{[r]}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT^*, (J^rT)^*)$ of operators*

$C_M^{[r]}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rT^*M, (J^rTM)^*), \quad C_M^{[r]}(g) = I_g^{[r]}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 9. *The family $\tilde{C}^{[r]}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT^*, (J^rT^*)^*)$ of operators*

$\tilde{C}_M^{[r]}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rT^*M, (J^rT^*M)^*), \quad \tilde{C}_M^{[r]}(g) = \tilde{I}_g^{[r]}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 10. *The family $D^{<r>}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT, (J^rT^*)^*)$ of operators*

$D_M^{<r>}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rTM, (J^rT^*M)^*), \quad D_M^{<r>}(g) = J_g^{<r>}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 11. *The family $\tilde{D}^{<r>}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT, (J^rT)^*)$ of operators*

$\tilde{D}_M^{<r>}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rTM, (J^rTM)^*), \quad \tilde{D}_M^{<r>}(g) = \tilde{J}_g^{<r>}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 12. *The family $D^{[r]}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT^*, (J^rT^*)^*)$ of operators*

$D_M^{[r]}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rT^*M, (J^rT^*M)^*), \quad D_M^{[r]}(g) = J_g^{[r]}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

Proposition 13. *The family $\tilde{D}^{[r]}: \mathbf{Riem} \rightsquigarrow \mathbf{Hom}(J^rT^*, (J^rT)^*)$ of operators*

$\tilde{D}_M^{[r]}: \mathbf{Riem}(M) \rightarrow \mathbf{Hom}_M(J^rT^*M, (J^rTM)^*), \quad \tilde{D}_M^{[r]}(g) = \tilde{J}_g^{[r]}$
for all $M \in \mathbf{obj}(\mathcal{M}f_m)$ is an $\mathcal{M}f_m$ -natural operator.

4. The main results. Let $g \in \text{Riem}(M)$ be a Riemannian metric on an m -manifold M . By Proposition 3 and Examples 1, 2, 3 we have identifications

$$\begin{aligned} J^r TM &= J^r T^* M = (J^r TM)^* = (J^r T^* M)^* = \bigoplus_{k=0}^r TM \otimes S^k T^* M \\ &= \bigoplus_{k=0}^r T^* M \otimes S^k T^* M = \bigoplus_{k=0}^r T^* M \otimes S^k TM = \bigoplus_{k=0}^r TM \otimes S^k TM \end{aligned}$$

modulo the base preserving vector bundle isomorphisms canonically depending on g .

Consequently, the problem of finding all $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, J^r T^*)$ is reduced to the one of finding all systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T^*, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): TM \otimes S^l T^* M \rightarrow T^* M \otimes S^k T^* M$, where $l, k = 1, \dots, r$ or (equivalently) our problem is reduced to the one of finding all natural tensors $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.

Thus we have proved the following theorem.

Theorem 1. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, J^r T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r TM \rightarrow J^r T^* M$ are in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

Because of the isomorphism $J^r TM \cong J^r T^* M$ depending on g , we have the following theorem.

Theorem 2. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, J^r T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r TM \rightarrow J^r TM$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): TM \otimes S^l T^* M \rightarrow TM \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T \otimes T \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

By the same reason, we have also the following corollary.

Corollary 1. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T^*, J^r T)$ transforming Riemannian metrics g on m -manifolds M into base preserving*

vector bundle maps $D_M(g): J^r T^* M \rightarrow J^r T M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l T^* M \rightarrow T M \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T \otimes T \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.

By the same reason, we have another corollary.

Corollary 2. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T^*, J^r T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r T^* M \rightarrow J^r T^* M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l T^* M \rightarrow T^* M \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

Because of the isomorphisms $J^r T M \cong J^r T^* M \cong (J^r T M)^*$ depending on g , we have the following theorem.

Theorem 3. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, (J^r T)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r T M \rightarrow (J^r T M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T^*, T^* \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T M \otimes S^l T^* M \rightarrow T^* M \otimes S^k T M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

By the same reason, we have the following theorem.

Theorem 4. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T^*, (J^r T)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r T^* M \rightarrow (J^r T M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T^*, T^* \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l T^* M \rightarrow T^* M \otimes S^k T M$ for*

$l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.

By the same reason, we have also the following corollary.

Corollary 3. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T)^*, J^r T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r TM)^* \rightarrow J^r TM$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l TM \rightarrow TM \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T^* \otimes T \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

We have also the next similar corollary.

Corollary 4. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T)^*, J^r T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r TM)^* \rightarrow J^r T^* M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l TM \rightarrow T^* M \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

We have also another corollary.

Corollary 5. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T)^*, (J^r T)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r TM)^* \rightarrow (J^r TM)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T, T^* \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l TM \rightarrow T^* M \otimes S^k TM$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T^* \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

Because of the isomorphisms $J^r TM \cong J^r T^* M \cong (J^r TM)^* \cong (J^r T^* M)^*$ depending on g , we have the following theorem.

Theorem 5. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T, (J^r T^*)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r T M \rightarrow (J^r T^* M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T^*, T \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T M \otimes S^l T^* M \rightarrow T M \otimes S^k T M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T \otimes T \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T \otimes T \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

By the same reason, we have the following theorem.

Theorem 6. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}(J^r T^*, (J^r T^*)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): J^r T^* M \rightarrow (J^r T^* M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T^*, T \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l T^* M \rightarrow T M \otimes S^k T M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T \otimes T \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T \otimes T \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

By the same reason, we have also the following theorem.

Theorem 7. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T)^*, (J^r T^*)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r T M)^* \rightarrow (J^r T^* M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T^* \otimes S^l T, T \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T^* M \otimes S^l T M \rightarrow T M \otimes S^k T M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T \otimes S^l T^* \otimes T \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T \otimes S^l T^* \otimes T \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

We have also the following corollary.

Corollary 6. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T^*)^*, J^r T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r T^* M)^* \rightarrow J^r T M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T, T \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): T M \otimes S^l T M \rightarrow T M \otimes S^k T^* M$ for $l, k = 1, \dots, r$*

or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T^* \otimes T \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T^* \otimes T \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.

We have also the next corollary.

Corollary 7. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T^*)^*, J^r T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r T^* M)^* \rightarrow J^r T^* M$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): TM \otimes S^l TM \rightarrow T^* M \otimes S^k T^* M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T^* \otimes T^* \otimes S^k T^*$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T^* \otimes T^* \otimes S^k T^*$ on M for $l, k = 1, \dots, r$.*

We have also the similar corollary.

Corollary 8. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T^*)^*, (J^r T)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r T^* M)^* \rightarrow (J^r T M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T, T^* \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): TM \otimes S^l TM \rightarrow T^* M \otimes S^k T M$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T^* \otimes T^* \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T^* \otimes T^* \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

Finally, we have the last corollary.

Corollary 9. *The $\mathcal{M}f_m$ -natural operators $D: \text{Riem} \rightsquigarrow \text{Hom}((J^r T^*)^*, (J^r T^*)^*)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $D_M(g): (J^r T^* M)^* \rightarrow (J^r T^* M)^*$ are in the bijection with systems $(A^{l,k})$ of $\mathcal{M}f_m$ -natural operators $A^{l,k}: \text{Riem} \rightsquigarrow (T \otimes S^l T, T \otimes S^k T)$ transforming Riemannian metrics g on m -manifolds M into base preserving vector bundle maps $A_M^{l,k}(g): TM \otimes S^l TM \rightarrow TM \otimes S^k TM$ for $l, k = 1, \dots, r$ or (equivalently) in the bijection with systems $(t^{l,k})$ of $\mathcal{M}f_m$ -natural operators (natural tensors) $t^{l,k}: \text{Riem} \rightsquigarrow T^* \otimes S^l T^* \otimes T \otimes S^k T$ transforming Riemannian metrics g on m -manifolds M into tensor fields of types $T^* \otimes S^l T^* \otimes T \otimes S^k T$ on M for $l, k = 1, \dots, r$.*

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